This is a previous version of the article published in Vietnam Journal of Mathematics. 2020, 48: 315-334. doi:10.1007/s10013-020-00402-6

Subdifferentials and Stability Analysis of Feasible Set and Pareto Front Mappings in Linear Multiobjective Optimization*

M. J. Cánovas† M. A. López‡ B. S. Mordukhovich§
J. Parra†

Abstract

The paper concerns multiobjective linear optimization problems in $\mathbb{R}^n$ that are parameterized with respect to the right-hand side perturbations of inequality constraints. Our focus is on measuring the variation of the feasible set and the Pareto front mappings around a nominal element while paying attention to some specific directions. This idea is formalized by means of the so-called epigraphical multifunction, which is defined by adding a fixed cone to the images of the original mapping. Through the epigraphical feasible and Pareto front mappings we describe the corresponding vector subdifferentials and employ them to verifying Lipschitzian stability of the perturbed mappings with computing the associated Lipschitz moduli. The particular case of ordinary linear programs is analyzed, where we show that the subdifferentials of both multifunctions are proportional subsets. We also provide a method for computing the optimal value of linear programs without knowing any optimal solution. Some illustrative examples are also given in the paper.

Key words. Epigraphical set-valued mappings, feasible set mappings, Lipschitz moduli, linear programming, optimal value functions, multiobjective optimization.

AMS Subject Classification: 49J53, 90C31, 15A39, 90C05, 90C29.

*This research has been partially supported by grants MTM2014-59179-C2-(1,2)-P and PGC2018-097960-B-C2(1,2).

†Center of Operations Research, Miguel Hernández University of Elche, 03202 Elche (Alicante), Spain (canovas@umh.es, parra@umh.es).

‡Department of Mathematics, University of Alicante, 03080 Alicante, Spain (marco.antonio@ua.es); CIAO, Federation University, Ballarat, Australia. Research of this author is also partially supported by the Australian Research Council (ARC) Discovery Grants Scheme (Project Grant # DP180100602).

§Department of Mathematics, Wayne State University, Detroit, MI 48202, USA (boris@math.wayne.edu). Research of this author was partially supported by the USA National Science Foundation under grants DMS-1512846 and DMS-1808978, by the USA Air Force Office of Scientific Research grant #15RT04, and by Australian Research Council under grant DP-190100555.

1 Introduction and Overview

The original motivation for this paper comes from analyzing Lipschitzian behavior of the so-called Pareto front mapping associated with the *multiobjective linear programming* (MLP) problem given by

$$\begin{array}{lll} MLP\,(b): & \textit{minimize} & (\langle c_1, x\rangle, ..., \langle c_q, x\rangle) \\ & \textit{subject to} & \quad x \in \mathcal{F}\,(b), \end{array} \tag{1}$$

where $x \in \mathbb{R}^n$ is the decision variable, where $c_1, ..., c_q \in \mathbb{R}^n$ are fixed, and where $\mathcal{F}\,(b)$ is the *feasible set* of the linear inequality system in $\mathbb{R}^n$ parameterized by its right-hand side (RHS) as

$$\sigma\,(b) := \left\{ \langle a_t, x\rangle \leq b_t,\ \ t \in T := \{1, ..., m\} \right\} \tag{2}$$

with the coefficients $a_t \in \mathbb{R}^n$ fixed for each $t \in T$ and the perturbation parameter $b = (b_t)_{t\in T} \in \mathbb{R}^T$ in the RHS of (2).

For each $b \in \mathbb{R}^T$ denote by $\mathcal{S}\,(b)$ the set of *nondominated solutions* to $MLP\,(b)$, i.e., $\mathcal{S}\,(b)$ is formed by all $x \in \mathcal{F}\,(b)$ such that there does not exist any other feasible point $y \in \mathcal{F}\,(b)$ for which $\langle c_i, y\rangle \leq \langle c_i, x\rangle$ whenever $i = 1, ..., q$ and $\langle c_{i_0}, y\rangle < \langle c_{i_0}, x\rangle$ for some $i_0 \in \{1, ..., q\}$. Alternatively it can be reformulated as follows: considering the mapping $\mathcal{C}\colon \mathbb{R}^n \to \mathbb{R}^q$ defined by $\mathcal{C}(x) := (\langle c_1, x\rangle, ..., \langle c_q, x\rangle)$, we have the equivalence

$$x \in \mathcal{S}\,(b) \ \Leftrightarrow\ \big(\mathcal{C}(\mathcal{F}\,(b) - x)\big) \cap \big(-\mathbb{R}^q_+\big) = \{0_q\}.$$

Associated with the parameterized problem (1), we define the *Pareto front mapping* $\mathcal{P}\colon \mathbb{R}^T \rightrightarrows \mathbb{R}^q$ by

$$\mathcal{P}\,(b) := \{(\langle c_1, x\rangle, ..., \langle c_q, x\rangle),\ x \in \mathcal{S}\,(b)\} = \mathcal{C}(\mathcal{S}\,(b)). \tag{3}$$

Observe that in the case of ordinary/scalar linear programming (LP) problem, i.e., when $q = 1$, the Pareto front mapping $\mathcal{P}$ reduces to the real-valued *optimal value function* known also as the 'marginal function' in variational analysis.

Appropriate tools of *variational analysis* and *generalized differentiation* are our primary machinery to study the major (robust) *Lipschitzian stability* notion for the feasible set and Pareto front mappings. To proceed, we need to compute the *subdifferential* of these set-valued mappings/multifunctions, which is defined via the *coderivative* of the corresponding epigraphical multifunctions; see Section 2. At this moment we advance that a natural definition of the *epigraphical Pareto front mapping* $\mathcal{E}_\mathcal{P}\colon \mathbb{R}^T \rightrightarrows \mathbb{R}^q$ is given by

$$\mathcal{E}_\mathcal{P}\,(b) := \mathcal{P}\,(b) + \mathbb{R}^q_+, \tag{4}$$

where $\mathbb{R}^q_+$ is formed by the elements of $\mathbb{R}^q$ with nonnegative components.

Roughly speaking, while analyzing optimality in MLP we are interested only in that region of the feasible set where optimal/nondominated solutions may be located. A possible idea to skip the noninteresting regions is to consider a certain epigraphical mapping associated with the feasible set mapping. In this way we define the *epigraphical feasible set mapping* $\mathcal{E}_{\mathcal{F}}\colon \mathbb{R}^T \rightrightarrows \mathbb{R}^n$ by

$$\mathcal{E}_{\mathcal{F}}\left(b\right) := \mathcal{F}\left(b\right) + \left\{c_1, ..., c_q\right\}^{\circ}, \tag{5}$$

where $\Omega^{\circ} := \{y \in \mathbb{R}^n | \, \langle y, x\rangle \geq 0 \;\text{ for all }\; x \in \mathbb{R}^n\}$ stands for the (positive) polar cone of the set $\Omega \subset \mathbb{R}^n$.

The *main contributions* of our paper are precise calculations of the subdifferentials of the set-valued mappings $\mathcal{F}$ and $\mathcal{P}$ with the subsequent usage of them to verify Lipschitzian stability of these mappings and computing the corresponding Lipschitz moduli by invoking the powerful machinery of variational analysis. We show below that the subdifferentials of these multifunctions and their Lipschitz moduli are closely related as seen in Theorems 7 and 8, and the established relationships are particularly clear in the case of ordinary (single-objective) linear programs; see Proposition 3 and Theorem 9.

Given a mapping $\mathcal{M}\colon Y \rightrightarrows Z$ between metric spaces Y and Z with the graph

$$\mathrm{gph}\mathcal{M} := \left\{(y,z) \in Y \times Z \middle| \; z \in \mathcal{M}(y\right\}$$

and with the same notation d for the metrics on Y and Z, its Lipschitzian behavior is analyzed locally around a fixed point $(\overline{y}, \overline{z}) \in \mathrm{gph}\mathcal{M}$ while reflecting the rate of variation of its images with respect to the variation of the corresponding preimages. Here we focus on the most natural graphical extension of the classical local Lipschitz continuity to set-valued mappings that is spread in variational analysis as the Lipschitz-like/pseudo-Lipschitz/Aubin property. For definiteness let us say that $\mathcal{M}$ is *Lipschitz-like* around $(\overline{y}, \overline{z}) \in \mathrm{gph}\mathcal{M}$ if there exist neighborhoods $U \subset Y$ and $V \subset Z$ of $\overline{y}$ and $\overline{z}$, respectively, and a constant $\ell \geq 0$ such that we have the linear estimate

$$d\big(z, \mathcal{M}\left(y'\right)\big) \leq \ell \, d\left(y, y'\right) \;\text{ for all }\; y, y' \in U \;\text{ and all }\; z \in V \cap \mathcal{M}\left(y\right). \tag{6}$$

Each constant ℓ ensuring (6) for associated neighborhoods U and V is called a Lipschitz constant and the infimum of such Lipschitz constants is called the *Lipschitz modulus*, or the *exact Lipschitz bound* of $\mathcal{M}$ around $(\overline{y}, \overline{z})$, and is denoted by $\mathrm{lip}\mathcal{M}\left(\overline{y}, \overline{z}\right)$. We can easily check that

$$\mathrm{lip}\mathcal{M}\left(\overline{y}, \overline{z}\right) = \limsup_{\substack{y, y' \to \overline{y} \\ z \to \overline{z}, z \in \mathcal{M}(y)}} \frac{d\big(z, \mathcal{M}\left(y'\right)\big)}{d\left(y, y'\right)} \tag{7}$$

under the convention that $0/0 := 0$. It has been well recognized in variational analysis that the Lipschitz-like property (6) and its inverse mapping equivalences known as *metric regularity* and *linear openness/covering* play a fundamental role in many aspects of optimization, equilibrium, systems control, and applications; see the monographs [3, 9, 13, 14, 16, 17, 19] and the references therein.

Using the modulus representation (7), we can rephrase that the main contribution of this paper is to *explicitly compute* the quantities $\mathrm{lip}\mathcal{E}_{\mathcal{F}}\left(\overline{b}, \overline{x}\right)$ and

$\text{lip}\mathcal{E}_{\mathcal{P}}\left(\overline{b},\overline{p}\right)$, with $\left(\overline{b},\overline{x}\right)\in \text{gph}\mathcal{E}_{\mathcal{F}}$ and $\left(\overline{b},\overline{p}\right)\in \text{gph}\mathcal{E}_{\mathcal{P}}$ respectively, entirely in terms of the given data of (1) and (2). Furthermore, we advance here that the number $\text{lip}\mathcal{E}_{\mathcal{P}}\left(\overline{b},\overline{p}\right)$ provides a lower estimate of $\text{lip}\mathcal{P}\left(\overline{b},\overline{p}\right)$, and that both Lipschitz moduli *agree* for ordinary *linear programs* as shown in Section 5. Having the precise formulas for computing the moduli $\text{lip}\mathcal{E}_{\mathcal{F}}\left(\overline{b},\overline{x}\right)$ and $\text{lip}\mathcal{E}_{\mathcal{P}}\left(\overline{b},\overline{p}\right)$, the *necessary and sufficient conditions* for *Lipschitzian stability* of the mappings (4) and (5)—in the sense of the validity of the Lipschitz-like property for these mappings around the reference points—are formulated now as, respectively,

$$\text{lip}\mathcal{E}_{\mathcal{F}}\left(\overline{b},\overline{x}\right)<\infty \;\text{ and }\; \text{lip}\mathcal{E}_{\mathcal{P}}\left(\overline{b},\overline{p}\right)<\infty.$$

These achievements are largely based on the *subdifferential* notion for *set-valued mappings* with *ordered values* introduced in [1] (see also [2, 17]) via the coderivatives concept for mappings and on the *coderivative criterion* for the Lipschitz-like property of multifunctions established in [15]. The passage from coderivatives to subdifferentials of ordered mappings was accomplished in [1] via the usage of *epigraphical multifunctions*: the pattern well understood in variational analysis for the subdifferential-coderivative relationship concerning scalar (extended-real-valued) functions; see, e.g., [16, Vol. 1, p. 84].

It is worth mentioning that some coderivative analysis of frontier and efficient solution mapping was provided in [12] for problems of vector optimization with respect to the so-called *generalized order optimality* (including Pareto efficiency) in infinite-dimensional spaces. However, neither precise coderivative formulas, nor subdifferential analysis, nor computations of Lipschitz moduli were obtained in the general setting of [12] in contrast to what is done in this paper.

Furthermore, while confining to the case of ordinary/scalar linear programs where $\mathcal{P}$ is the optimal value function, the reader is addressed to [10] for different formulas concerning Lipschitz moduli in various parametric frameworks. Note also that Lipschitzian behavior of the 'ordinary' feasible set mapping $\mathcal{F}$ and the computation of its modulus were derived for more general models of semi-infinite and infinite programming in [4] and [7]. Other stability properties of the feasible set mapping of linear semi-infinite systems were analyzed in [11, Chapter 6]. Lipschitzian behavior of the optimal set, again in the context of linear programming problems (in fact, in a continuous convex semi-infinite setting allowing also perturbations of the objective function) was studied in [6], whereas the associated Lipschitz modulus was computed in [5].

The rest of the paper is organized as follows. In Section 2 we present the necessary notation, definitions, and results about coderivatives, subdifferentials, and Lipschitz moduli that are needed later on. Section 3 is devoted to subdifferential analysis and Lipschitzian stability of the epigraphical feasible set mappings $\mathcal{E}_{\mathcal{F}}$ from (5). Specifically, we provide explicit descriptions of the subdifferential of $\mathcal{F}$ and the Lipschitz modulus of $\mathcal{E}_{\mathcal{F}}$ at a given point of its graph. In the subsequent Section 4 we develop a constructive procedure for deriving the representation of such a mapping as the feasible set mapping associated with new parameterized systems of linear programming. Section 5 is focussed on the precise computations of subdifferential and Lipschitz modulus of the epigraphical Pareto front mapping $\mathcal{E}_{\mathcal{P}}$ from (4). In Section 6 we consider the case of ordinary linear programs (with only one objective function) and show that even in this case our results are new. In particular, we establish exact relationships between the subdifferentials and Lipschitz moduli of the set-valued mappings $\mathcal{E}_{\mathcal{F}}$ and $\mathcal{E}_{\mathcal{P}}$ under consideration. Both Sections 5 and 6 contain illustrative examples of their own interest. The final Section 7 summarizes the obtained results and discusses some directions of future research.

Throughout the paper we use the standard notion in variational analysis and optimization. Recall that $\operatorname{conv}\Omega$, $\operatorname{cone}\Omega$, and $\operatorname{span}\Omega$ stand, respectively, for the *convex hull*, the *conic convex hull*, and the *linear subspace* generated by the set $\Omega \subset \mathbb{R}^n$ under the convention that $\operatorname{cone}\emptyset = \{0_n\}$, where 0_n is the origin of $\mathbb{R}^n$. If Ω is convex, by $O^+(\Omega)$ we represent the *recession cone* of Ω. The space of decision variables $\mathbb{R}^n$ is endowed with an arbitrary norm $\|\cdot\|$, while the space of parameters $\mathbb{R}^T$ is equipped with the supremum norm

$$\|b\|_\infty := \sup_{t\in T} |b_t| . \tag{8}$$

2 Preliminaries and First Results

In this section, unless otherwise stated, $\mathcal{M}\colon Y \rightrightarrows Z$ is a set-valued mapping between Banach spaces Y and Z which topological duals are denoted by Y^* and Z^*, respectively. The *coderivative* of $\mathcal{M}$ at $(\overline{y},\overline{z}) \in \operatorname{gph}\mathcal{M}$ is a positively homogeneous multifunction $D^*\mathcal{M}(\overline{y},\overline{z}) : Z^* \rightrightarrows Y^*$ defined by

$$y^* \in D^*\mathcal{M}(\overline{y},\overline{z})(z^*) \Longleftrightarrow (y^*,-z^*) \in N\big((\overline{y},\overline{z});\operatorname{gph}\mathcal{M}\big), \tag{9}$$

where $N((\overline{y},\overline{z});\operatorname{gph}\mathcal{M})$ is the (basic, limiting, Mordukhovich) *normal cone* to $\operatorname{gph}\mathcal{M}$ at $(\overline{y},\overline{z})$; see, e.g., [16] and [19]. For simplicity, $\|\cdot\|$ stands for the norm in any Banach space X, and $\|\cdot\|_*$ is the corresponding dual norm, i.e.,

$$\|x^*\|_* = \sup\big\{ \langle x^*,x\rangle \;\big|\; \|x\| \le 1,\ x \in X\big\},\ x^* \in X^*,$$

where $\langle .,. \rangle$ denotes the canonical pairing between X and X^*. If no confusion arises, from now on we skip the subscript '$_*$' in the dual norm notation.

When both spaces Y and Z are finite-dimensional and the graph of $\mathcal{M}$ is locally closed around $(\overline{y},\overline{z}) \in \operatorname{gph}\mathcal{M}$, there is the following *precise formula* for the *computing the Lipschitz modulus* of $\mathcal{M}(\overline{y},\overline{z})$:

$$\operatorname{lip}\mathcal{M}(\overline{y},\overline{z}) = \|D^*\mathcal{M}(\overline{y},\overline{z})\| := \sup\big\{ \|y^*\|_* \,\big|\, y^* \in D^*\mathcal{M}(\overline{y},\overline{z})(z^*), \|z^*\|_* = 1\big\}, \tag{10}$$

which was obtained in [15]. We also refer the reader to [19, Theorem 9.40] for another proof of this result, which was labeled therein as the Mordukhovich criterion. An infinite-dimensional extension of (10) was derived in [16, Theorem 4.10]. It is more involved and is not used in this paper dealing with finite-dimensional multiobjective optimization problems of type (1). A simplified proof of (10) in finite dimensions was given in [17, Theorem 3.3].

If the graph of $\mathcal{M}$ is *convex*, the normal cone in (9) reduces to the normal convex of convex analysis, and hence $y^* \in D^*\mathcal{M}(\overline{y}, \overline{z})(z^*)$ if and only if

$$\langle (y^*, -z^*), (y' - \overline{y}, z' - \overline{z}) \rangle \le 0 \ \text{ for all } \ (y', z') \in \mathrm{gph}\mathcal{M},$$

which is equivalent to the description

$$\langle y^*, y' - \overline{y} \rangle \le \langle z^*, z' - \overline{z} \rangle \quad \text{for all} \quad (y', z') \in \mathrm{gph}\mathcal{M}. \tag{11}$$

Given further a closed and convex *ordering cone* $\Theta \subset Z$, the *epigraphical multifunction* $\mathcal{E}_{\mathcal{M}} \colon X \rightrightarrows Z$ associated with $\mathcal{M}$ and the cone Θ is that which graph $\mathrm{gph}\mathcal{E}_{\mathcal{M}}$ coincides with the *epigraph* of $\mathcal{M}$ with respect to Θ. In other words, we have $\mathcal{E}_{\mathcal{M}}(y) := \mathcal{M}(y) + \Theta$ and

$$\mathrm{epi}\mathcal{M} := \mathrm{gph}\mathcal{E}_{\mathcal{M}} = \left\{ (y, z) \,\middle|\, z \in \mathcal{M}(y) + \Theta \right\},$$

where we skip indicating Θ in the epigraphical notation.

In accordance with [1], we present the following definition of the subdifferential of $\mathcal{M}$ at the reference point of its epigraph with respect to Θ.

Definition 1 *Let* $(\overline{y}, \overline{z}) \in \mathrm{epi}\mathcal{M}$ *be given. The* SUBDIFFERENTIAL *of* $\mathcal{M}$ *at* $(\overline{y}, \overline{z})$ *denoted as* $\partial\mathcal{M}(\overline{y}, \overline{z})$ *is a subset of* Y^* *defined by*

$$\partial\mathcal{M}(\overline{y}, \overline{z}) := \left\{ y^* \in D^*\mathcal{E}_{\mathcal{M}}(\overline{y}, \overline{z})(z^*) \,\middle|\, -z^* \in N(0; \Theta),\ \|z^*\|_* = 1 \right\}, \tag{12}$$

where $N(0; \Theta) \subset Z^*$ *is the convex normal cone to the set* Θ *at the origin of* Z*.*

Note that if $\mathcal{M} \colon Y \to \mathbb{R}$ is a proper convex function with $\Theta = \mathbb{R}_+$, then $\mathrm{gph}\mathcal{E}_{\mathcal{M}}$ is its standard epigraph, and for any $\overline{y} \in \mathrm{dom}\,\mathcal{M}$ the set $\partial\mathcal{M}(\overline{y}, \mathcal{M}(\overline{z}))$ is the classical subdifferential of $\mathcal{M}$ at $\overline{y}$ in the sense of convex analysis.

Observe also that the set $-N(0; \Theta)$ is nothing else but the polar cone Θ°, and thus we have the following representation of the coderivative of $\mathcal{E}_{\mathcal{M}}$ in terms of the graph $\mathrm{gph}\mathcal{M}$ instead of the epigraph $\mathrm{epi}\mathcal{M}$.

Proposition 1 *Assume that* $\mathrm{epi}\mathcal{M}$ *is a convex set, and let* $(\overline{y}, \overline{z}) \in \mathrm{epi}\mathcal{M}$*. Then for any* $z^* \in Z^*$ *we have the representation*

$$\begin{aligned} & D^*\mathcal{E}_{\mathcal{M}}(\overline{y}, \overline{z})(z^*) \\ & = \begin{cases} \{ y^* \in Y^* \mid \langle y^*, y' - \overline{y} \rangle \le \langle z^*, z' - \overline{z} \rangle \ \ \forall (y', z') \in \mathrm{gph}\mathcal{M} \} & \text{if } \ z^* \in \Theta^\circ, \\ \emptyset \ \ \text{if } \ z^* \notin \Theta^\circ. \end{cases} \end{aligned}$$

Proof. Take $z^* \in \Theta^\circ$. By using the definitions of the coderivative (9) and of the epigraphical multifunction $\mathcal{E}_{\mathcal{M}}$, we get due to the convexity of $\mathrm{epi}\mathcal{M}$ ($= \mathrm{gph}\mathcal{E}_{\mathcal{M}}$) that

$$D^*\mathcal{E}_{\mathcal{M}}(\overline{y}, \overline{z})(z^*) = \left\{ y^* \mid \langle y^*, y' - \overline{y} \rangle \le \langle z^*, z' - \overline{z} \rangle \ \forall (y', z') \in \mathrm{gph}\mathcal{E}_{\mathcal{M}} \right\}. \tag{13}$$

Let us show that $\mathrm{gph}\mathcal{E}_{\mathcal{M}}$ can be equivalently replaced by $\mathrm{gph}\mathcal{M}$ in (13). Indeed, take any $y^* \in Y^*$ satisfying (11). Pick further any $(\widetilde{y}, \widetilde{z}) \in \mathrm{gph}\mathcal{E}_{\mathcal{M}}$ and write $\widetilde{z} = z' + u$ with $z' \in \mathcal{M}(\widetilde{y})$ and $u \in \Theta$. Then we obtain the inequalities

$$\langle y^*, \widetilde{y} - \overline{y} \rangle \leq \langle z^*, z' - \overline{z} \rangle \leq \langle z^*, \widetilde{z} - \overline{z} \rangle$$

due to $\langle z^*, u \rangle \geq 0$. It gives us the claimed coderivative formula for $z^* \in \Theta^\circ$.

Suppose now that $z^* \notin \Theta^\circ$ and find $u \in \Theta$ such that $\langle z^*, u \rangle < 0$. Arguing by contradiction, assume that there is $y^* \in D^*\mathcal{E}_{\mathcal{M}}(\overline{y}, \overline{z})(z^*)$, i.e., by (11) we have

$$\langle y^*, y' - \overline{y} \rangle \leq \langle z^*, z' - \overline{z} \rangle \quad \text{for all} \quad (y', z') \in \mathrm{gph}\mathcal{E}_{\mathcal{M}}.$$

Since $\mathcal{E}_{\mathcal{M}}(\overline{y}) + \Theta = \mathcal{E}_{\mathcal{M}}(\overline{y})$, it follows that $(\overline{y}, \overline{z} + u) \in \mathrm{gph}\mathcal{E}_{\mathcal{M}}$, and therefore

$$0 = \langle y^*, \overline{y} - \overline{y} \rangle \leq \langle z^*, u \rangle,$$

which is a contradiction that completes the proof of the proposition. ■

Employing Proposition 1 leads us to deriving effective representations of the subdifferential of $\mathcal{M}$ and the Lipschitz modulus of $\mathcal{E}_{\mathcal{M}}$ as well as to a relation between the latter and the Lipschitz modulus of $\mathcal{M}$ at the reference point.

Theorem 1 *Let the epigraphical set* $\mathrm{epi}\mathcal{M}$ *be convex, and let* $(\overline{y}, \overline{z}) \in \mathrm{epi}\mathcal{M}$. *Then we have the subdifferential representation*

$$\partial\mathcal{M}(\overline{y}, \overline{z}) = \bigcup_{\substack{z^* \in \Theta^\circ \\ \|z^*\|_* = 1}} \left\{ y^* \middle| \ \langle y^*, y' - \overline{y} \rangle \leq \langle z^*, z' - \overline{z} \rangle \, \forall (y', z') \in \mathrm{gph}\mathcal{M} \right\}. \quad (14)$$

If in addition Y and Z are finite-dimensional and if the set $\mathrm{epi}\mathcal{M}$ *is locally closed around* $(\overline{y}, \overline{z})$, *then the Lipschitz modulus of* $\mathcal{E}_{\mathcal{M}}$ *at* $(\overline{y}, \overline{z})$ *is computed by*

$$\mathrm{lip}\mathcal{E}_{\mathcal{M}}(\overline{y}, \overline{z}) = \sup \left\{ \|y^*\|_* \ \middle| \ y^* \in \partial\mathcal{M}(\overline{y}, \overline{z}) \right\}. \quad (15)$$

Assuming furthermore that $(\overline{y}, \overline{z}) \in \mathrm{gph}\mathcal{M}$ *and that the set* $\mathrm{gph}\mathcal{M}$ *is locally closed around this point, we conclude that*

$$\mathrm{lip}\mathcal{E}_{\mathcal{M}}(\overline{y}, \overline{z}) \leq \mathrm{lip}\mathcal{M}(\overline{y}, \overline{z}). \quad (16)$$

Proof. Representation (14) follows directly from definition (12) of the subdifferential $\partial\mathcal{M}(\overline{y}, \overline{z})$ combined with Proposition 1.

Assuming now that the spaces Y and Z are finite-dimensional, applying the Lipschitz modulus formula (10) to the epigraphical mapping $\mathcal{E}_{\mathcal{M}}$, and appealing again to Proposition 1 tell us that

$$\begin{aligned} \mathrm{lip}\mathcal{E}_{\mathcal{M}}(\overline{y}, \overline{z}) &= \sup \left\{ \|y^*\|_* \ \middle| \ y^* \in D^*\mathcal{E}_{\mathcal{M}}(\overline{y}, \overline{z})(z^*), \ \|z^*\|_* = 1 \right\} \\ &= \sup \left\{ \|y^*\|_* \ \middle| \ y^* \in D^*\mathcal{E}_{\mathcal{M}}(\overline{y}, \overline{z})(z^*), \ z^* \in \Theta^\circ, \ \|z^*\|_* = 1 \right\}. \end{aligned}$$

Thus the claimed formula (15) follows from the definition of $\partial\mathcal{M}(\overline{y}, \overline{z})$.

To verify finally the inequality (16), denote by $\widehat{N}\left(\left(\overline{y},\overline{z}\right);\Omega\right)$ the prenormal/regular normal cone to $\Omega\subset Y\times Z$ at $\left(\overline{y},\overline{z}\right)$ (see, e.g., [16, 19]) and using the convexity of epi$\mathcal{M}$, we get

$$N\left(\left(\overline{y},\overline{z}\right);\mathrm{epi}\mathcal{M}\right)=\widehat{N}\left(\left(\overline{y},\overline{z}\right);\mathrm{epi}\mathcal{M}\right)\subset\widehat{N}\left(\left(\overline{y},\overline{z}\right);\mathrm{gph}\mathcal{M}\right)=N\left(\left(\overline{y},\overline{z}\right);\mathrm{gph}\mathcal{M}\right),$$

where the inclusion comes from [16, Proposition 1.5]. This gives us

$$D^{*}\mathcal{E}_{\mathcal{M}}\left(\overline{y},\overline{z}\right)(z)\subset D^{*}\mathcal{M}\left(\overline{y},\overline{z}\right)(z)\quad\text{for all}\quad z\in Z$$

and thus deduces (16) from the basic coderivative formula (10). ■

Remark 1 *The inequality in* (16) *may be* strict *as illustrated by the following simple example. Consider* $\Theta:=\mathbb{R}_{+}$ *and* $\mathcal{M}\colon\mathbb{R}\rightrightarrows\mathbb{R}$ *given by* $\mathcal{M}(y):=[y,2y]$ *if* $y\geq0$ *and* $\mathcal{M}(y):=\{0\}$ *if* $y<0$*. Then it is easy to calculate that*

$$1=\mathrm{lip}\mathcal{E}_{\mathcal{M}}\left(0,0\right)<\mathrm{lip}\mathcal{M}\left(0,0\right)=2.$$

3 Stability Analysis of Epigraphical Feasible Sets

The underlying goal of this section is explicit computing the Lipschitz modulus of epigraphical feasible set mapping associated with the parameterized MLP problem (1). As we know from Sections 1 and 2, our approach reduces this computation to deriving a verifiable formula to calculate the subdifferential in the sense of Definition 1 of the perturbed feasible set $\mathcal{F}$ in terms of its given data. Proceeding in this way, we concentrate here on obtaining the representations of the subdifferential and Lipschitz modulus with involving the graph of the nondominated solution mapping $\mathcal{S}$.

Let us begin with two lemmas. The first one is a well-known result that gives a characterization of nondominated solutions to MLP(b) via optimal solutions to a scalarized linear program. We formulate it without a proof. The second lemma is a new result, which plays a key role throughout the paper.

Lemma 1 *Let* $x_0\in\mathcal{F}(b)$ *for some* $b\in\mathbb{R}^{T}$*. Then the following are equivalent:*
(i) $x_0\in\mathcal{S}(b)$*.*
(ii) *There exist numbers* $\lambda_i>0$ *for* $i=1,...,q$*, such that*

$$x_0\in\arg\min\left\{\sum_{i=1}^{q}\lambda_i\left\langle c_i,x\right\rangle\;\middle|\;x\in\mathcal{F}(b)\right\}.$$

To formulate the second lemma, recall that

$$\mathrm{dom}\mathcal{S}:=\left\{b\in\mathbb{R}^{T}\,\middle|\,\mathcal{S}(b)\neq\emptyset\right\}.$$

Lemma 2 *Let* $b\in\mathrm{dom}\mathcal{S}$*. Then for any* $x_0\in\mathcal{F}(b)\backslash\mathcal{S}(b)$ *there exists* $\widetilde{x}_0\in\mathcal{S}(b)$ *such that* $\langle c_i,\widetilde{x}_0\rangle\leq\langle c_i,x_0\rangle$ *whenever* $i=1,...,q$*.*

Proof. Fix $x_0 \in \mathcal{F}(b) \setminus \mathcal{S}(b)$ and proceed step-by-step as follows:

Step 1. Let us proof the existence of solutions to the linear program:

$$x_1 \in \arg\min \big\{ \langle c_1, x \rangle \,\big|\, x \in \mathcal{F}(b), \;\; \langle c_i, x \rangle \le \langle c_i, x_0 \rangle, \;\; i = 1, ..., q \big\}. \tag{17}$$

Arguing by contradiction, suppose that (17) has no optimal solutions. Since x_0 is a feasible solution to (17), our assumption is equivalent to the unboundedness of the set of feasible solutions to the linear program (17). Thus there exists a sequence $\{w_r\}_{r \in \mathbb{N}} \subset \mathbb{R}^n$ such that

$$w_r \in \mathcal{F}(b), \;\; \langle c_i, w_r \rangle \le \langle c_i, x_0 \rangle, \;\; i = 1, ..., q, \;\text{ for all } r \in \mathbb{N},$$

while we have the infinite limit

$$\lim_{r \to \infty} \langle c_1, w_r \rangle = -\infty.$$

Remembering that $\mathcal{S}(b) \neq \emptyset$, pick any $\widetilde{x} \in \mathcal{S}(b)$ and find by Lemma 1 numbers $\lambda_i > 0$ with $i = 1, ..., q$ such that

$$\widetilde{x} \in \arg\min \left\{ \sum_{i=1}^{q} \lambda_i \langle c_i, x \rangle \,\middle|\, x \in \mathcal{F}(b) \right\}.$$

This readily brings us to the contradiction:

$$\sum_{i=1}^{q} \lambda_i \langle c_i, \widetilde{x} \rangle \le \sum_{i=1}^{q} \lambda_i \langle c_i, w_r \rangle \le \lambda_1 \langle c_1, w_r \rangle + \sum_{i=2}^{q} \lambda_i \langle c_i, x_0 \rangle \underset{r \to \infty}{\longrightarrow} -\infty,$$

which therefore verifies the existence of the solution x_1 to (17). Note furthermore that if x_1 satisfies $x_1 \in \mathcal{S}(b)$, then the proof of the lemma is complete. Otherwise we go to the next step as follows.

Step 2. Suppose that $x_1 \in \mathcal{F}(b) \setminus \mathcal{S}(b)$. Then arguing as in Step 1 ensures the existence of a vector $x_2 \in \mathbb{R}^n$ satisfying

$$x_2 \in \arg\min \big\{ \langle c_2, x \rangle \,\big|\, x \in \mathcal{F}(b), \;\; \langle c_i, x \rangle \le \langle c_i, x_1 \rangle, \;\; i = 1, ..., q \big\}. \tag{18}$$

As before, the proof of the lemma is finished if $x_2 \in \mathcal{S}(b)$. Otherwise we go to *Step* 3 and proceed similarly.

Reaching in this way *Step* j with some $j < q$, we either finish the proof, or arrive at *Step* q that is described below.

Step q. Suppose that $x_{q-1} \in \mathcal{F}(b) \setminus \mathcal{S}(b)$. Again we get

$$x_q \in \arg\min \big\{ \langle c_q, x \rangle \,\big|\, x \in \mathcal{F}(b), \;\; \langle c_i, x \rangle \le \langle c_i, x_{q-1} \rangle, \;\; i = 1, ..., q \big\}.$$

Let us show that now we do not have any choice but $x_q \in \mathcal{S}(b)$. Arguing by contradiction, assume that there exists $w \in \mathcal{F}(b)$ such that

$$\begin{cases} \langle c_i, w \rangle \le \langle c_i, x_q \rangle \;\text{ for all }\; i = 1, ..., q, \\ \langle c_j, w \rangle < \langle c_j, x_q \rangle \;\text{ for some }\; j \in \{1, ..., q\}. \end{cases}$$

Then we arrive at a contradiction with the choice of x_j. Indeed, it follows that

$$\begin{array}{rcl} \langle c_i, w\rangle & \leq & \langle c_i, x_q\rangle \leq \langle c_i, x_{q-1}\rangle \leq ... \leq \langle c_i, x_{j-1}\rangle \text{ for all } i = 1, ..., q, \\ \langle c_j, w\rangle & < & \langle c_j, x_q\rangle \leq \langle c_j, x_{q-1}\rangle \leq ... \leq \langle c_j, x_j\rangle . \end{array}$$

This completes the proof of the lemma. ■

The next theorem provides a description of the subdifferential $\partial\mathcal{F}\left(\overline{b}, \overline{x}\right)$ in terms of gph$\mathcal{S}$ (instead of gph$\mathcal{F}$ as in the definition), which eventually allows us to relate the subdifferential $\partial\mathcal{F}\left(\overline{b}, \overline{x}\right)$ to the subdifferential of the Pareto front mapping (3). This leads us to new results even in the case of standard linear programs as shown in Section 6.

Remark 2 Using the notation of Section 2 gives us

$$\mathcal{E}_{\mathcal{F}}\left(b\right) = \mathcal{F}\left(b\right) + \Theta \text{ for all } b \in \mathbb{R}^T \text{ with } \Theta := \left\{c_1, ..., c_q\right\}^{\circ}.$$

From now on we denote

$$C := -N\left(0_n; \Theta\right) = \Theta^{\circ} = \operatorname{cone}\left\{c_1, ...c_q\right\},$$

where the last equality immediately follows from the classical Farkas Lemma.

Here is the aforementioned theorem with the subdifferential calculation. In the paper, and despite $\mathbb{R}^n$ is self-dual, we are using $\|c\|_*$ and $\|a_t\|_*$ because c and a_t are regarded as linear functions ($x \mapsto \langle c, x\rangle$ and $x \mapsto \langle a_t, x\rangle$, respectively).

Theorem 2 *Let* $\left(\overline{b}, \overline{x}\right) \in \operatorname{gph}\mathcal{E}_{\mathcal{F}}$. *Then we have the subdifferential formula*

$$\partial\mathcal{F}\left(\overline{b}, \overline{x}\right) = \bigcup_{\substack{c \in C \\ \|c\|_* = 1}} \left\{y \in \mathbb{R}^T \middle| \left\langle y, b - \overline{b}\right\rangle \leq \langle c, x - \overline{x}\rangle \text{ for all } (b, x) \in \operatorname{gph}\mathcal{S}\right\}. \tag{19}$$

Proof. By the convexity of the sets gph$\mathcal{F}$ and gph$\mathcal{E}_{\mathcal{F}}$ we get from (14) that

$$\partial\mathcal{F}\left(\overline{b}, \overline{x}\right) = \bigcup_{\substack{c \in C \\ \|c\|_* = 1}} \left\{y \in \mathbb{R}^T \middle| \left\langle y, b - \overline{b}\right\rangle \leq \langle c, x - \overline{x}\rangle \text{ for all } (b, x) \in \operatorname{gph}\mathcal{F}\right\}.$$

Since gph$\mathcal{S}$ $\subset$gph$\mathcal{F}$, we only need to verify the inclusion '$\supset$' of (19).

To proceed, pick any $c \in C$ with $\|c\|_* = 1$ and select $y \in \mathbb{R}^T$ such that

$$\left\langle y, b - \overline{b}\right\rangle \leq \langle c, x - \overline{x}\rangle \text{ for all } (b, x) \in \operatorname{gph}\mathcal{S}. \tag{20}$$

Arguing by contradiction, suppose that there exists $(b_0, x_0) \in \operatorname{gph}\mathcal{F}$ with

$$\left\langle y, b_0 - \overline{b}\right\rangle > \langle c, x_0 - \overline{x}\rangle ,$$

which yields $(b_0, x_0) \notin \text{gph}\mathcal{S}$. Applying then Lemma 2 to (b_0, x_0) ensures the existence of $\widetilde{x}_0 \in \mathcal{S}(b_0)$ such that $\langle c_i, \widetilde{x}_0 \rangle \leq \langle c_i, x_0 \rangle$ for all $i = 1, ..., q$. In particular, we get $\langle c, \widetilde{x}_0 \rangle \leq \langle c, x_0 \rangle$. Therefore

$$\left\langle y, b_0 - \overline{b} \right\rangle > \langle c, x_0 - \overline{x} \rangle \geq \langle c, \widetilde{x}_0 - \overline{x} \rangle ,$$

which contradicts (20) and thus completes the proof of the theorem. ■

Now we are ready to establish a precise formula for computing the Lipschitz modulus at $\left(\overline{b}, \overline{x}\right) \in \text{gph}\mathcal{E}_{\mathcal{F}}$ of the epigraphical feasible set mapping from (5). In the next theorem we employ the l_1-norm $\|\cdot\|_1$ on $\mathbb{R}^T$, which is dual to the primal supremum norm (8) used above.

Theorem 3 *Let $\left(\overline{b}, \overline{x}\right) \in \text{gph}\mathcal{E}_{\mathcal{F}}$. Then we have*

$$\begin{aligned} \text{lip}\mathcal{E}_{\mathcal{F}}\left(\overline{b}, \overline{x}\right) &= \sup\left\{ \|y\|_1 \,\middle|\, y \in \partial\mathcal{F}\left(\overline{b}, \overline{x}\right) \right\} \\ &= \sup\Big\{ \bigcup_{\substack{c \in C \\ \|c\|_* = 1}} \left\{ \|y\|_1 \,\middle|\, \left\langle y, b - \overline{b} \right\rangle \leq \langle c, x - \overline{x} \rangle \;\; \forall\, (b, x) \in \text{gph}\mathcal{S} \right\}\Big\}, \end{aligned}$$

and thus the multifunction $\mathcal{E}_{\mathcal{F}}$ is Lipschitz-like around $\left(\overline{b}, \overline{x}\right)$ if and only if

$$\sup\Big\{ \bigcup_{\substack{c \in C \\ \|c\|_* = 1}} \left\{ \|y\|_1 \,\middle|\, \left\langle y, b - \overline{b} \right\rangle \leq \langle c, x - \overline{x} \rangle \;\; \forall \;\, (b, x) \in \text{gph}\mathcal{S} \right\}\Big\} < \infty.$$

Proof. Observe that $(b, x) \in \text{gph}\mathcal{F} \iff a_t' x - e_t' b \leq 0$ for all $t = 1, ..., m$, where $e_t \in \mathbb{R}^m$ is the t-th vector of the canonical basis of $\mathbb{R}^m$. Appealing to Remark 2, we see that the set $\text{gph}\mathcal{E}_{\mathcal{F}}$ is a polyhedral convex cone admitting the representation

$$\text{gph}\mathcal{E}_{\mathcal{F}} = \text{gph}\mathcal{F} + \left(\{0_m\} \times \left\{c_1, ..., c_q\right\}^\circ\right);$$

so this set is closed and convex. Thus the claimed modulus formula follows from (15) and Theorem 2. The last statement of this theorem follows directly from the definition of the Lipschitz modulus and the formula for its computation. ■

Examples 1 and 2 illustrate both Theorem 2 and Theorem 3. They are included in the next section for comparative purposes, specifically to point out the similarities between the subdifferentials $\partial\mathcal{F}$ and $\partial\mathcal{P}$.

4 Computation Formulas for Feasible Sets

In this section we derive a precise formula for representing the epigraphical multifunction $\mathcal{E}_{\mathcal{F}}$ from (5) via solutions of a new linear inequality system associated with $\mathcal{F}(b)$. More constructive representations are obtained for some specific forms of feasible solution sets that are especially important for applications. All of this constitutes, in particular, the basis for computations of the optimal value in linear programs, which is illustrated and further developed in Section 6 in the framework of Example 3.

Let us start revealing the following relationship between the 'multiobjective epigraphical feasible set mapping' $\mathcal{E}_{\mathcal{F}}$ and its linear program counterpart $\mathcal{F}(b)+\{c\}^{\circ}$ coming from a certain scalarization technique.

Theorem 4 *For any $b \in \mathbb{R}^T$ we have the relationship*

$$\mathcal{E}_{\mathcal{F}}(b) = \bigcap_{c \in \operatorname{conv}\{c_1,...,c_q\}} \left(\mathcal{F}(b) + \{c\}^{\circ}\right).$$

Proof. Confining ourselves to the nontrivial case where $\mathcal{F}(b) \neq \emptyset$, observe first that the inclusion '$\subset$' follows from the obvious fact that

$$\{c_1,...,c_q\}^{\circ} = \bigcap_{c \in \operatorname{conv}\{c_1,...,c_q\}} \{c\}^{\circ}.$$

To verify the opposite inclusion '$\supset$', assume that $x \notin \mathcal{E}_{\mathcal{F}}(b)$ and then show that there exists $c \in \operatorname{conv}\{c_1,...,c_q\}$ such that $x \notin \mathcal{F}(b) + \{c\}^{\circ}$. Denote by $\widehat{x}$ the Euclidean projection of x onto $\mathcal{E}_{\mathcal{F}}(b)$. It is well known that

$$\langle \widehat{x} - x, y\rangle \geq \langle \widehat{x} - x, \widehat{x}\rangle \text{ for all } y \in \mathcal{E}_{\mathcal{F}}(b). \tag{21}$$

In particular, for any $y_0 \in \mathcal{F}(b)$, all $u \in \{c_1,...,c_q\}^{\circ}$, and all $\lambda > 0$ we have $\langle \widehat{x} - x, y_0 + \lambda u\rangle \geq \langle \widehat{x} - x, \widehat{x}\rangle$. Dividing both sides of the latter inequality by $\lambda > 0$ and letting $\lambda \to \infty$ give us $\langle \widehat{x} - x, u\rangle \geq 0$, i.e.,

$$\widehat{x} - x \in \{c_1,...,c_q\}^{\circ\circ} = \operatorname{cone}\{c_1,...,c_q\}.$$

Thus we have $\widehat{x} - x = \mu c$ for some $c \in \operatorname{conv}\{c_1,...,c_q\}$ and some $\mu > 0$ by taking into account that $\widehat{x} - x \neq 0_n$. To verify now that $x \notin \mathcal{F}(b) + \{c\}^{\circ}$, suppose the contrary and then deduce from the above that $x = y + u$ with some $y \in \mathcal{F}(b)$ and $u \in \{c\}^{\circ}$. It tells us that

$$\langle c, x\rangle = \langle c, y\rangle + \langle c, u\rangle \geq \langle c, y\rangle \geq \langle c, \widehat{x}\rangle > \langle c, x\rangle,$$

where the penultimate step comes from (21), while the last one follows from the projection inequality

$$\langle c, \widehat{x} - x\rangle = \frac{1}{\mu}\|\widehat{x} - x\|_2^2$$

with $\|\cdot\|_2$ standing for the Euclidean norm. The obtained contradiction completes the proof of the theorem. ■

Remark 3 Observe that in Theorem 4 we cannot avoid the convex combination in the representation of $\mathcal{E}_{\mathcal{F}}(b)$, i.e., replace $\operatorname{conv}\{c_1,...,c_q\}$ by $\{c_1,...,c_q\}$. To illustrate it, consider the case where $\mathbb{R}^T = \mathbb{R}^2$,

$$\mathcal{F}(b) = \operatorname{conv}\{(1,0),(0,1)\}, \text{ and } c_1 = (1,0), \; c_2 = (0,1).$$

However, the set $\operatorname{conv}\{c_1,...,c_q\}$ can be replaced by any basis of the cone C.

To establish efficient representations of the sets in the form $\mathcal{F}(b) + \{c\}^{\circ}$, and hence of $\mathcal{E}_{\mathcal{F}}(b)$ due to Theorem 4, we focus now on multifunctions $\mathcal{F}$ defined as

$$\mathcal{F}(\cdot) + \text{cone}\{u\} \quad \text{and} \quad \mathcal{F}(\cdot) + \text{span}\{u\}.$$

This is done in the remainder of this section.

Given $u \in \mathbb{R}^n$, consider first the polyhedral set $\mathcal{F}(b)$+cone $\{u\}$ and introduce the following partition $\{T_1, T_2\}$ of $T = \{1, ..., m\}$:

$$T_1 := \left\{t \in T \middle| \ \langle a_t, u \rangle \leq 0\right\} \ \text{ and } \ T_2 := \left\{t \in T \middle| \ \langle a_t, u \rangle > 0\right\}. \tag{22}$$

Then for each $t \in T_1$ we denote

$$a_{(t,0)} := a_t \text{ and } b_{(t,0)} := b_t$$

and for each $(t, s) \in T_1 \times T_2$ denote

$$a_{(t,s)} := \langle a_s, u \rangle \, a_t - \langle a_t, u \rangle \, a_s, \quad b_{(t,s)} := \langle a_s, u \rangle \, b_t - \langle a_t, u \rangle \, b_s. \tag{23}$$

With $\widetilde{T} := T_1 \times (\{0\} \cup T_2)$ let us now define the linear inequality system

$$\widetilde{\sigma}(b) := \left\{ \left\langle a_{(t,s)}, x \right\rangle \leq b_{(t,s)}, \ \ (t,s) \in \widetilde{T} \right\} \tag{24}$$

and denote by $\widetilde{\mathcal{F}}(b)$ the set of feasible solutions to $\widetilde{\sigma}(b)$.

Remark 4 If $T_1 = \emptyset$*. then* $\widetilde{T} = \emptyset$ *and* $\widetilde{\mathcal{F}}(b) = \mathbb{R}^n$*. Otherwise we have that* $\left(a_{(t,s)}, b_{(t,s)}\right)$ *is a conic combination of* (a_t, b_t) *and* (a_s, b_s) *for all* $(t, s) \in \widetilde{T}$*. It is clear then that* $\mathcal{F}(b) \subset \widetilde{\mathcal{F}}(b)$*. Observe also that, in contrast to* $\sigma(b)$*, the new system* $\widetilde{\sigma}(b)$ *is no longer parameterized by its RHS.*

The next theorem represents $\mathcal{F}(b)$ + cone $\{u\}$ as the set of feasible solutions to the new linear inequality system (24).

Theorem 5 *In terms of the notation above, for any* $b \in \mathbb{R}^T$ *we have*

$$\widetilde{\mathcal{F}}(b) = \mathcal{F}(b) + \text{cone}\ \{u\}. \tag{25}$$

Proof. Let us first verify the inclusion '$\supset$' in (25). Taking $x \in \mathcal{F}(b)$+cone $\{u\}$, we get the linear inequalities

$$\langle a_t, x - \mu u \rangle \leq b_t \text{ for all } \ t \in T \text{ and some } \ \mu \geq 0. \tag{26}$$

There is nothing to prove if $T_1 = \emptyset$. Otherwise we fix $t \in T_1$ and get $\langle a_t, x \rangle \leq b_t$. Taking further $s \in T_2$, we distinguish the following two cases. If $\langle a_t, u \rangle = 0$, then the aimed inequality

$$\left\langle a_{(t,s)}, x \right\rangle \leq b_{(t,s)}$$

reduces to $\langle a_t, x\rangle \le b_t$. In the case where $\langle a_t, u\rangle < 0$ we deduce from (26) that

$$\frac{\langle a_s, x\rangle - b_s}{\langle a_s, u\rangle} \le \mu \le \frac{\langle a_t, x\rangle - b_t}{\langle a_t, u\rangle}.$$

In particular, it follows that

$$\frac{\langle a_s, x\rangle - b_s}{\langle a_s, u\rangle} \le \frac{\langle a_t, x\rangle - b_t}{\langle a_t, u\rangle},$$

which readily implies that

$$\left\langle a_{(t,s)}, x\right\rangle \le b_{(t,s)}$$

and thus verifies the inclusion '$\supset$' in (25).

To prove now the opposite inequality '$\subset$' in (25), pick any $x \in \widetilde{\mathcal{F}}(b)$ and let us verify the existence of $\mu \ge 0$ such that

$$\langle a_t, x - \mu u\rangle \le b_t \ \text{ for all } \ t \in T.$$

Indeed, when $T_2 = \emptyset$ we get $x \in \mathcal{F}(b)$, which agrees in this case with $\widetilde{\mathcal{F}}(b)$. If $T_2 \neq \emptyset$, it is sufficient to consider any $\mu \ge 0$ satisfying

$$\max_{s\in T_2} \frac{\langle a_s, x\rangle - b_s}{\langle a_s, u\rangle} \le \mu \le \inf_{\langle a_t, u\rangle < 0} \frac{\langle a_t, x\rangle - b_t}{\langle a_t, u\rangle}$$

under the usual convention that $\inf \emptyset = \infty$. We complete the proof of the theorem by observing that such a number μ exists due to the choice of x. ■

Looking closely at the proof of Theorem 5 tells us that the successive application of the procedure therein is instrumental to represent the more general feasible sets $\mathcal{F}(b) + \text{cone}\,\{u_1, ..., u_p\}$ via linear inequality systems. However, explicit forms of such representations may generally be rather complicated. In the next theorem we consider the important case where

$$\mathcal{F}(b) + \text{span}\left\{u\right\}$$

for which we give a direct proof.

Theorem 6 *Given any $b \in \mathbb{R}^T$ and recalling the notation in* (23)*, we have*

$$\mathcal{F}(b) + \text{span}\big\{u\big\} = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \langle a_t, x\rangle \le b_t \ \textit{if} \ \langle a_t, u\rangle = 0, \\ \left\langle a_{(t,s)}, x\right\rangle \le b_{(t,s)} \ \textit{if} \ \langle a_t u\rangle < 0 \\ \qquad\qquad\qquad \textit{and} \ \langle a_s, u\rangle > 0 \end{array} \right\}. \quad (27)$$

Proof. Let us introduce the index set

$$T_0 := \{t \in T \mid \langle a_t, u\rangle = 0\}(\subset T_1).$$

Note that, in the case $T_1 = \emptyset$ (or $T_2 = T_0 = \emptyset$), the system (27) has no inequality (i.e., its solution set is the whole space $\mathbb{R}^n$), but in such cases $\mathcal{F}(b) + \text{span}\ \{u\}$

is also $\mathbb{R}^n$, as $-u \in \operatorname{int}(O^+\mathcal{F}(b))$ (or $u \in \operatorname{int}(O^+\mathcal{F}(b))$, respectively), entailing that if $x_0 \in \mathcal{F}(b)$,

$$x_0 + \operatorname{span}\{u\} + \lambda\mathbb{B} \subset \mathcal{F}(b) + \operatorname{span}\{u\}, \text{ for all } \lambda \geq 0.$$

If $T_1 \diagdown T_0$ and T_2 are both nonempty, the reasoning is the same followed in Proposition 5, without taking into account the sign of μ. ■

The reader will see in Example 3 below a detailed illustration of both Theorems 5 and 6 together with additional comments on the relationship between the optimal value and the epigraphical mapping $\mathcal{E}_{\mathcal{F}}$ in linear programming.

5 Subdifferentials of Epigraphical Pareto Fronts

This section concerns the epigraphical Pareto front multifunction $\mathcal{E}_{\mathcal{P}} \colon \mathbb{R}^T \rightrightarrows \mathbb{R}^q$ introduced in (4) in the form

$$\mathcal{E}_{\mathcal{P}}(b) := \mathcal{P}(b) + \mathbb{R}^q_+, \quad b \in \mathbb{R}^T,$$

where the Pareto front mapping $\mathcal{P}$ is defined in (3). In contrast to $\operatorname{gph}\mathcal{F}$, the set $\operatorname{gph}\mathcal{P}$ is nonconvex in general, while the one of our interest $\operatorname{gph}\mathcal{E}_{\mathcal{P}}$ is *always convex*. This is shown in the next proposition.

Proposition 2 *The set* $\operatorname{gph}\mathcal{E}_{\mathcal{P}}$ *is a closed and convex subset of* $\mathbb{R}^T \times \mathbb{R}^q$.

Proof. First we observe that the set $\operatorname{gph}\mathcal{P}$ is a finite union of convex polyhedral cones as the KKT (or primal/dual) optimality conditions in linear programming allow us to express $\operatorname{gph}\mathcal{P}$ as the graph of a certain feasible set mapping of a linear system and we can apply then the classical result by Robinson [18]. Hence *a fortiori* the set $\operatorname{gph}\mathcal{E}_{\mathcal{P}} = \operatorname{gph}\mathcal{P} + \left(\{0_m\} \times \mathbb{R}^q_+\right)$ is also closed.

Let us now show that the set $\operatorname{gph}\mathcal{E}_{\mathcal{P}}$ is convex. Fix any two pairs $(b_1, p_1 + u_1)$, $(b_2, p_2 + u_2) \in \operatorname{gph}\mathcal{E}_{\mathcal{P}}$, i.e., such that $b_i \in \mathbb{R}^T$, $p_i \in \mathcal{P}(b_i)$, and $u_i \in \mathbb{R}^q_+$ as $i = 1, 2$. Then for every $\lambda \in [0, 1]$ we have

$$p_i = (\langle c_1, x_i \rangle, ..., \langle c_q, x_i \rangle) \text{ with some } x_i \in \mathcal{S}(b_i), \ i = 1, 2,$$

and so $(1-\lambda)x_1 + \lambda x_2 \in \mathcal{F}((1-\lambda)b_1 + \lambda b_2)$. In the nontrivial case where

$$(1-\lambda)x_1 + \lambda x_2 \notin \mathcal{S}((1-\lambda)b_1 + \lambda b_2)$$

we apply Lemma 2 to get the existence of $\widetilde{x} \in \mathcal{S}((1-\lambda)b_1 + \lambda b_2)$ with $\langle c_i, \widetilde{x} \rangle \leq \langle c_i, (1-\lambda)x_1 + \lambda x_2 \rangle$ for all $c_1, \ldots, c_q$. It implies that

$$(1-\lambda)p_1 + \lambda p_2 \in (\langle c_1, \widetilde{x} \rangle, ..., \langle c_q, \widetilde{x} \rangle) + \mathbb{R}^q_+,$$

which can be equivalently written as

$$(1-\lambda)p_1 + \lambda p_2 \in \mathcal{E}_{\mathcal{P}}((1-\lambda)b_1 + \lambda b_2).$$

Therefore, we arrived at the inclusion

$$(1-\lambda)(p_1+u_1)+\lambda(p_2+u_2)\in\mathcal{E}_{\mathcal{P}}((1-\lambda)b_1+\lambda b_2),$$

which verifies the convexity of the set $\mathrm{gph}\mathcal{E}_{\mathcal{P}}$. ■

Using the above proposition and employing the fundamental results of Theorem 1, we can now conduct a local stability analysis of the epigraphical Pareto front mapping similarly to that for the epigraphical feasible solution mapping developed in Section 3.

Theorem 7 *Let $(\overline{b},\overline{p})\in\mathrm{gph}\mathcal{P}$. Then we have*

$$\partial\mathcal{P}(\overline{b},\overline{p})=\bigcup_{\substack{\alpha\in\mathbb{R}^q_+\\ \|\alpha\|_*=1}}\left\{y\in\mathbb{R}^T\,\middle|\,\left\langle y,b-\overline{b}\right\rangle\le\langle\alpha,p-\overline{p}\rangle\ \text{for all}\ (b,p)\in\mathrm{gph}\mathcal{P}\right\}.$$

Furthermore, the Lipschitz modulus of the epigraphical Pareto front mapping $\mathcal{E}_{\mathcal{P}}$ at $(\overline{b},\overline{x})$ is computed by the formula

$$\mathrm{lip}\mathcal{E}_{\mathcal{P}}(\overline{b},\overline{x})=\sup\left\{\{\|y\|_1\,\middle|\,y\in\partial\mathcal{P}(\overline{b},\overline{p})\right\},$$

which ensures that the mapping $\mathcal{E}_{\mathcal{P}}$ is Lipschitz-like around $(\overline{b},\overline{p})$ if and only if

$$\sup\left\{\{\|y\|_1\,\middle|\,\bigcup_{\substack{\alpha\in\mathbb{R}^q_+\\ \|\alpha\|_*=1}}\left\{y\in\mathbb{R}^T\,\middle|\,\left\langle y,b-\overline{b}\right\rangle\le\langle\alpha,p-\overline{p}\rangle\,\text{for all}\ (b,p)\in\mathrm{gph}\mathcal{P}\right\}\right\}<\infty.$$

Proof. Having in hand Proposition 2, we can apply Theorem 1 and then proceed similarly to the proofs of Theorems 1 and 3. ■

The next result expresses the subdifferential $\partial\mathcal{F}(\overline{b},\overline{x})$ in terms of $\mathrm{gph}\mathcal{P}$ instead of $\mathrm{gph}\mathcal{S}$. Observe that the difference between the expression for $\partial\mathcal{F}(\overline{b},\overline{x})$ obtained below and the one for $\partial\mathcal{P}(\overline{b},\overline{p})$ established in Theorem 7 is seen only in the sets where the vector $\alpha\in\mathbb{R}^q_+$ takes its values.

Theorem 8 *Let $(\overline{b},\overline{x})\in\mathrm{gph}\mathcal{S}$, and let $\overline{p}=(\langle c_1,\overline{x}\rangle,...,\langle c_q,\overline{x}\rangle)\in\mathcal{P}(\overline{b})$. Then the subdifferential of $\mathcal{F}$ at $(\overline{b},\overline{x})$ is computed by*

$$\partial\mathcal{F}(\overline{b},\overline{x})=\bigcup_{\substack{\alpha\in\mathbb{R}^q_+\\ \left\|\sum\alpha_ic_i\right\|_*=1}}\left\{y\in\mathbb{R}^T\,\middle|\,\left\langle y,b-\overline{b}\right\rangle\le\langle\alpha,p-\overline{p}\rangle\ \text{for all}\ (b,p)\in\mathrm{gph}\mathcal{P}\right\}.$$

Proof. Taking into account the previous considerations, we proceed similarly to the proof of Theorem 2. ■

Let us now present a two-dimensional numerical example that illustrates how both Theorems 7 and 8 can be applied in computation.

Example 1 Take any $a > 0$ and consider the following multiobjective problem (1) with $n = q = 2$ and the Euclidean norms on both spaces*:*

$$
\begin{array}{lll}
MLP\,(b): & \textit{minimize} & (ax_1, x_2) \\
 & \textit{subject to} & x_1 \geq b_1, \\
 & & x_2 \geq b_2.
\end{array}
$$

Letting $\overline{b} = 0_2$*,* we easily see that

$$
\begin{array}{l}
\mathcal{F}\,(b) = \mathcal{E}_{\mathcal{F}}\,(b) = \{(b_1, b_2)\} + \mathbb{R}^2_+,\ \mathcal{S}\,(b) = \ \{(b_1, b_2)\}, \\
\mathcal{P}\,(b) = \ \{(ab_1, b_2)\},\ \textit{and}\ \mathcal{E}_{\mathcal{P}}\,(b) = \{(ab_1, b_2)\} + \mathbb{R}^2_+.
\end{array}
$$

Furthermore, using Theorems 2 and 7 tells us, respectively, that

$$
\begin{aligned}
\partial\mathcal{F}\,(0_2, 0_2) &= \bigcup_{\substack{\alpha^2\alpha_1^2+\alpha_2^2=1 \\ \alpha_1,\alpha_2\geq 0}} \left\{y \in \mathbb{R}^2 \middle|\ b_1y_1 + b_2y_2 \leq a\alpha_1 b_1 + \alpha_2 b_2 \ \ \forall\, b_1, b_2 \in \mathbb{R}\right\} \\
&= \left\{y \in \mathbb{R}^2 \middle|\ y_1^2 + y_2^2 = 1,\ y_1, y_2 \geq 0\right\},
\end{aligned}
$$

$$
\begin{aligned}
\partial\mathcal{P}\,(0_2, 0_2) &= \bigcup_{\substack{\alpha_1^2+\alpha_2^2=1 \\ \alpha_1,\alpha_2\geq 0}} \left\{y \in \mathbb{R}^2 \middle|\ b_1y_1 + b_2y_2 \leq a\alpha_1 b_1 + \alpha_2 b_2\ \forall\, b_1, b_2 \in \mathbb{R}\right\} \\
&= \left\{(y_1, y_2) \in \mathbb{R}^2 \middle|\ \frac{y_1^2}{a^2} + y_2^2 = 1,\ y_1, y_2 \geq 0\right\}.
\end{aligned}
$$

Since $\mathcal{F} = \mathcal{E}_{\mathcal{F}}$ in this case, we can appeal to Theorem 3 (cf. also [4, Corollary 3.2]) to compute the Lipschitz modulus:

$$
\mathrm{lip}\mathcal{F}\,(0_2, 0_2) = \mathrm{lip}\mathcal{E}_{\mathcal{F}}\,(0_2, 0_2) = \sqrt{2}.
$$

Considering now the mappings $\mathcal{P}$ and $\mathcal{E}_{\mathcal{P}}$*,* we can treat them as the feasible set mappings for the equality and inequality systems with respect to the variables (p_1, p_2). Namely, as $(1/a)p_1 = b_1, p_2 = b_2$ and $(1/a)p_1 \geq b_1, p_2 \geq b_2$*,* respectively. Appealing to Theorem 7 (cf. also [4, Corollary 3.2]), we obtain

$$
\mathrm{lip}\mathcal{P}\,(0_2, 0_2) = \mathrm{lip}\mathcal{E}_{\mathcal{P}}\,(0_2, 0_2) = \sqrt{1 + a^2}.
$$

As follows from (16), we have

$$
\mathrm{lip}\mathcal{P}\left(\overline{b}, \overline{p}\right) \geq \mathrm{lip}\mathcal{E}_{\mathcal{P}}\left(\overline{b}, \overline{p}\right),\ \text{for any}\ \left(\overline{b}, \overline{p}\right) \in \mathrm{gph}\mathcal{P}. \tag{28}
$$

The next example shows that the inequality in (28) may be *strict.*

Example 2 Consider the multiobjective problem (1) with $n = q = 2$ and the Euclidean norms on both spaces*:*

$$\begin{array}{lll} MLP\,(b): & \textit{minimize} & (x_1, x_2) \\ & \textit{subject to} & x_1 \geq b_1, \\ & & x_1 + x_2 \geq b_2. \end{array}$$

It is easy to check that $\mathcal{F}\,(b) = \mathcal{E}_{\mathcal{F}}\,(b)$ *for any* $b \in \mathbb{R}^2$ and that

$$\mathrm{lip}\mathcal{F}\,(0_2, 0_2) = \mathrm{lip}\mathcal{E}_{\mathcal{F}}\,(0_2, 0_2) = 1.$$

On the other hand, we clearly have the expressions

$$\begin{aligned} \mathcal{P}\,(b) &= \left\{p \in \mathbb{R}^2 \middle|\, p_1 \geq b_1, p_1 + p_2 = b_2\right\},\ b \in \mathbb{R}^2, \\ \mathcal{E}_{\mathcal{P}}\,(b) &= \left\{p \in \mathbb{R}^2 \middle|\, p_1 \geq b_1, p_1 + p_2 \geq b_2\right\},\ b \in \mathbb{R}^2, \end{aligned}$$

with the strict inequality

$$\mathrm{lip}\mathcal{P}\,(0_2, 0_2) = \sqrt{5} > \mathrm{lip}\mathcal{E}_{\mathcal{P}}\,(0_2, 0_2) = 1.$$

Observe that the situation of Example 2 does not occur in the case of single-objective linear programs, where we always have $\mathrm{lip}\mathcal{P}\left(\overline{b}, \overline{p}\right) = \mathrm{lip}\mathcal{E}_{\mathcal{P}}\left(\overline{b}, \overline{p}\right)$ with $\left(\overline{b}, \overline{p}\right) \in \mathrm{gph}\mathcal{P}$. This is one of the main points of the next section.

6 Lipschitz Moduli in Linear Programming

In this section we provide specifications and further developments of the results obtained above for the general linear multiobjective problem (1) for the case of ordinary linear programs given by

$$\begin{array}{lll} LP\,(b): & \textit{minimize} & \langle c, x\rangle \\ & \textit{subject to} & x \in \mathcal{F}\,(b) \end{array} \tag{29}$$

where the vector $c \in \mathbb{R}^n$ is fixed. As shown below, the approach and results developed for linear multiobjective problems lead us to refined computation formulas for subdifferentials and Lipschitz moduli in standard problems of linear programming under parameter perturbations.

In what follows we assume that $-c \in \mathrm{cone}\{a_t, t \in T\}$ (dual consistency); otherwise the problem $LP\,(b)$ is always unsolvable. Denote by $\vartheta\colon \mathbb{R}^T \to \mathbb{R} \cup \{+\infty\}$ the associated *optimal value function* defined by

$$\vartheta\,(b) := \inf\left\{\, \langle c, x\rangle \,\middle|\, x \in \mathcal{F}\,(b)\,\right\}.$$

We can easily see in this framework that

$$\mathrm{dom}\vartheta := \left\{b \in \mathbb{R}^T \middle|\, \vartheta\,(b) < \infty\right) = \mathrm{dom}\mathcal{F}.$$

Indeed, it is well known in linear programming that the boundedness of $LP(b)$ is equivalent to its solvability, which is in turn equivalent to the simultaneous fulfilment of primal and dual consistency.

Observe that in the setting of (29) the multifunctions $\mathcal{S}$, $\mathcal{P}$, $\mathcal{E}_{\mathcal{P}}$, and $\mathcal{E}_{\mathcal{F}}$ admit the following specifications. For each $b \in \mathbb{R}^T$ we have that $\mathcal{S}(b)$ is the set of optimal solutions to $LP(b)$ while the mapping $\mathcal{P}\colon \mathbb{R}^T \rightrightarrows \mathbb{R}$ is actually single-valued given by

$$\mathcal{P}(b) = \begin{cases} \{\vartheta(b)\} \text{ if } b \in \mathrm{dom}\mathcal{S}, \\ \emptyset \text{ if } b \notin \mathrm{dom}\mathcal{S}. \end{cases}$$

Furthermore, we get the relationships

$$\mathrm{dom}\mathcal{P} = \mathrm{dom}\mathcal{S} = \mathrm{dom}\mathcal{F} = \mathrm{dom}\vartheta.$$

Taking into account that $\mathcal{P}(b)$ is a singleton for any $b \in \mathrm{dom}\mathcal{S}$, from now on we write $\partial\mathcal{P}(b)$ instead of $\partial\mathcal{P}(b, \vartheta(b))$. Moreover, for each such b it follows that

$$\mathcal{E}_{\mathcal{P}}(b) = [\vartheta(b), \infty),$$

and it is easily to verify that

$$\mathcal{E}_{\mathcal{F}}(b) := \mathcal{F}(b) + \{c\}^{\circ} = \left\{x \in \mathbb{R}^n \middle|\ \langle c, x\rangle \geq \vartheta(b)\right\}. \tag{30}$$

It allows us to show below that Lipschitzian behavior of $\mathcal{E}_{\mathcal{F}}$ is closely related to that of $\mathcal{P}$; see Theorem 9. To proceed, we first present the following proposition.

Proposition 3 *For any* $(\overline{b}, \overline{x}) \in \mathrm{gph}\mathcal{S}$ *we have*

$$\partial\mathcal{P}(\overline{b}) = \|c\|_* \,\partial\mathcal{F}(\overline{b}, \overline{x}).$$

Proof. It follows from Theorems 7 and 8 that

$$\begin{aligned} \partial\mathcal{P}(\overline{b}) & \left\{y \in \mathbb{R}^T \middle|\ \langle y, b - \overline{b}\rangle \leq p - \overline{p}\ \forall (b, p) \in \mathrm{gph}\mathcal{P}\right\} \\ &= \|c\|_* \left\{y \in \mathbb{R}^T \middle|\ \langle y, b - \overline{b}\rangle \leq \frac{1}{\|c\|_*}(p - \overline{p})\ \ \forall (b, p) \in \mathrm{gph}\mathcal{P}\right\} \\ &= \|c\|_* \,\partial\mathcal{F}(\overline{b}\overline{x}), \end{aligned}$$

which therefore verifies the claimed equality. ■

Remark 5 Given $\overline{b} \in \mathrm{dom}\mathcal{P}$ and remembering that $\mathcal{P}(b)$ is a singleton for any $b \in \mathrm{dom}\mathcal{P}$, we can write

$$\partial\mathcal{P}(\overline{b}) = \left\{y \in \mathbb{R}^T \middle|\ \langle y, b - \overline{b}\rangle \leq \vartheta(b) - \vartheta(\overline{b})\ \ \forall\, b \in \mathrm{dom}\mathcal{P}\right\},$$

which agrees with the classical subdifferential of $\mathcal{P}$ at $\overline{b}$ in the sense of convex analysis. It is actually not surprising since the convexity of the function ϑ can be clearly derived from Proposition 2. Going a little further, observe that the set $\mathrm{dom}\mathcal{F}$ can be replaced by any intersection of the form $\mathrm{dom}\mathcal{F} \cap \mathcal{U}_{\overline{b}}$, where $\mathcal{U}_{\overline{b}} \in \mathbb{R}^T$ is an arbitrary neighborhood of $\overline{b}$.

Now we are ready to formulate and prove the last theorem of this paper.

Theorem 9 *Let* $\left(\overline{b},\overline{x}\right) \in \mathrm{gph}\mathcal{S}$. *Then*

$$\mathrm{lip}\mathcal{P}\left(\overline{b}\right) = \mathrm{lip}\mathcal{E}_{\mathcal{P}}\left(\overline{b}\right) = \|c\|_* \,\mathrm{lip}\mathcal{E}_{\mathcal{F}}\left(\overline{b},\overline{x}\right).$$

Proof. The first equality is standard since $\mathcal{P}$ is single-valued on $\mathbb{R}$. The second equality follows from Theorem 7 and Proposition 3 with taking into account the fact that the Lipschitz moduli under consideration agree with the suprema of the norms in the corresponding subdifferentials. ■

The next example shows how the obtained results are applied in the case of two-dimensional linear programming with multiply inequality constraints.

Example 3 Consider the following parameterized linear program in the space $\mathbb{R}^2$ with the Euclidean norm on it*:*

$$\begin{array}{lll} PL\,(b): & \textit{minimize} & 2x_1 + x_2 \\ & \textit{subject to} & -x_1 - x_2 \le b_1 \\ & & -x_1 + 2x_2 \le b_2 \\ & & -2x_1 \le b_3 \\ & & 3x_1 + x_2 \le b_4 \end{array}$$

around the nominal parameter $\overline{b} = (-2, 1, -2, 7)$. Since

$$\mathcal{F}\,(b) + \{c\}^{\circ} = \mathcal{F}\,(b) + \mathrm{cone}\,\{(2,1)\} + \mathrm{span}\,\{(-1,2)\},$$

we first apply Theorem 5 with $u = (2,1)$. Recalling the notation above tells us that $T_1 = \{1,2,3\}$, $T_2 = \{4\}$, and

$$\widetilde{\sigma}\,(b) = \left\{ \begin{array}{l} -x_1 - x_2 \le b_1, \ \ -x_1 + 2x_2 \le b_2, \ \ -2x_1 \le b_3, \\ 2x_1 - 4x_2 \le 7b_1 + 3b_4, \ \ -7x_1 + 14x_2 \le 7b_2, \\ -2x_1 + 4x_2 \le 7b_3 + 4b_4 \end{array} \right\}.$$

Note that the fifth constraint is equivalent to the second one for all $b \in \mathbb{R}^{\widetilde{T}}$, while the sixth constraint is redundant at $\overline{b}$ but not at any b. Remark 5 implies that the sixth inequality is irrelevant in a local analysis around $\overline{b}$. Anyway, let us remove just the fifth inequality and renumber the resulting $\widetilde{T}$ as $\{1,...,5\}$. Then apply Theorem 6 to the reduced and renumbered system $\widetilde{\sigma}$ with $\widetilde{u} = (-1,2)$ to obtain $\langle \widetilde{a}_t, \widetilde{u}\rangle < 0$ for $t \in \{1,4\}$ and $\langle \widetilde{a}_s, \widetilde{u}\rangle > 0$ for $s \in \{2,3,5\}$. It gives us

$$\widetilde{\widetilde{\sigma}}\,(b) = \left\{ \begin{array}{c} -6x_1 - 3x_2 \le 5b_1 + b_2, \ \ -4x_1 - 2x_2 \le 2b_1 + b_3, \\ -12x_1 - 6x_2 \le 10b_1 + 7b_3 + 4b_4, \ 0 \le 35b_1 + 10b_2 + 15b_4, \\ -16x_1 - 8x_2 \le 14b_1 + 10b_3 + 6b_4, \ 0 \le 70b_1 + 70b_3 + 70b_4 \end{array} \right\}.$$

Hence for any $b \in \mathbb{R}^4$ the system $\widetilde{\widetilde{\sigma}}\,(b)$ is equivalent to the single inequality

$$2x_1 + x_2 \ge \max\left\{ -\frac{5b_1 + b_2}{3}, -\frac{2b_1 + b_3}{2}, -\frac{10b_1 + 7b_3 + 4b_4}{6}, -\frac{7b_1 + 5b_3 + 3b_4}{4} \right\}$$

provided that $\min\{7b_1+2b_2+3b_4, b_1+b_3+b_4\} \geq 0,$ while otherwise the system $\tilde{\tilde{\sigma}}(b)$ is infeasible. Since the right-hand side in $\tilde{\tilde{\sigma}}(b)$ is $(-9,-6,-6,45,-6,210)$, for any b close to $\overline{b}$ we have $\tilde{\tilde{\sigma}}(b) \equiv 2x_1+x_2 \geq 3$ and

$$\tilde{\tilde{\sigma}}(b) \equiv 2x_1+x_2 \geq \max\left\{-\frac{5b_1+b_2}{3}, -\frac{2b_1+b_3}{2}\right\}.$$

This readily implies for such b that

$$\mathcal{P}(b) = \max\left\{-\frac{5b_1+b_2}{3}, -\frac{2b_1+b_3}{2}\right\}.$$

Employing now Remark 5 and the classical formula of convex analysis for subdifferentiation of maximum functions gives us

$$\partial\mathcal{P}\left(\overline{b}\right) = \operatorname{conv}\big\{(-5/3,-1/3,0,0),(-1,0,-1/2,0)\big\}.$$

Then we deduce from Theorem 7 that

$$\operatorname{lip}\mathcal{E}_{\mathcal{P}}\left(\overline{b}\right) = \|(-5/3,-1/3,0,0)\|_1 = 2,$$

which ensures in turn by using Theorem 9 that

$$\operatorname{lip}\mathcal{P}\left(\overline{b}\right) = 2 \text{ and } \operatorname{lip}\mathcal{E}_{\mathcal{F}}\left(\overline{b},\overline{x}\right) = 2/\sqrt{5}$$

at any optimal solution $\overline{x}$ of $PL\left(\overline{b}\right)$.

Remark 6 Paper [10] provides an alternative way to compute the Lipschitz modulus $\operatorname{lip}\mathcal{P}\left(\overline{b}\right)$ under the additional assumption that at least one optimal solution $\overline{x}$ of $PL\left(\overline{b}\right)$ is known. As we see, the procedure described in Example 3 does not require such an *a priori* information.

7 Concluding Remarks

This paper demonstrates that employing appropriate tools of variational analysis and generalized differentiation of set-valued mappings allows us to efficiently deal with major sensitivity characteristics of perturbed linear multiobjective optimization problems. Namely, in this way we explicitly computed the subdifferentials of the feasible set and Pareto front mappings in such problems together with the exact moduli of their Lipschitzian stability.

In future research we plan to extend the variational approach and results obtained in this paper to *convex* problems of multiobjective optimization by reducing them to linear systems with *block perturbations*. Observe that a similar procedure has been explored for feasibility mappings in *semi-infinite* programming with both decision and parameter variables living in Banach spaces.

References

[1] T. Q. BAO and B. S. MORDUKHOVICH, *Variational principles for set-valued mappings with applications to multiobjective optimization*, Control and Cybernetics **36** (2007), 531–562.

[2] T. Q. BAO and B. S. MORDUKHOVICH, *Relative Pareto minimizers for multiobjective problems: existence and optimality conditions*, Math. Program. **122** (2010), 301–347.

[3] J. M. BORWEIN and Q. J. ZHU, *Techniques of Variational Analysis*, Springer, New York, 2005.

[4] M. J. CÁNOVAS, A. L. DONTCHEV, M. A. LÓPEZ and J. PARRA, *Metric regularity of semi-infinite constraint systems*, Math. Program. **104** (2005), 329–346.

[5] M. J. CÁNOVAS, F. J. GÓMEZ-SENENT and J. PARRA, *On the Lipschitz modulus of the argmin mapping in linear semi-infinite optimization*, Set-Valued Anal. **16** (2008), 511–538.

[6] M. J. CÁNOVAS, D. KLATTE, M. A. LÓPEZ and J. PARRA, *Metric regularity in convex semi-infinite optimization under canonical perturbations*, SIAM J. Optim **18** (2007), 717–732.

[7] M. J. CÁNOVAS, M. A. LÓPEZ, B. S. MORDUKHOVICH and J. PARRA, *Variational analysis in semi-infinite and infinite programming, I: Stability of linear inequality systems of feasible solutions*, SIAM J. Optim. **20** (2009), 1504–1526.

[8] M. J. CÁNOVAS, M. A. LÓPEZ, B. S. MORDUKHOVICH and J. PARRA, Quantitative stability of linear infinite inequality systems under block perturbations with applications to convex systems, *TOP* **20** (2012), 310–327.

[9] A. L. DONTCHEV and R. T. ROCKAFELLAR, *Implicit Functions and Solution Mappings: A View from Variational Analysis*, 2nd edition, Springer, New York, 2014.

[10] M. J. GISBERT, M. J. CÁNOVAS, J. PARRA and F. J. TOLEDO, *Lipschitz modulus of the optimal value in linear programming*, J. Optim. Theory Appl. **182** (2019), 133–152.

[11] M. A. GOBERNA and M. A. LÓPEZ, *Linear Semi-Infinite Optimization*, John Wiley & Sons, Chichester, UK, 1998.

[12] N. Q. HUY, B. S. MORDUKHOVICH and J. C. YAO, *Coderivatives of frontier and solution maps in parametric multiobjective optimization*, Taiwanese J. Math. **12** (2008), 2083–2111.

[13] A. D. IOFFE, *Variational Analysis of Regular Mappings*, Springer, Cham, Switzerland, 2017.

[14] D. KLATTE and B. KUMMER, *Nonsmooth Equations in Optimization: Regularity, Calculus, Methods and Applications*, Kluwer Academic, Dordrecht, The Netherlands, 2002.

[15] B. S. MORDUKHOVICH, *Complete characterizations of openness, metric regularity, and Lipschitzian properties of multifunctions*, Trans. Amer. Math. Soc. **340** (1993), 1–35.

[16] B. S. MORDUKHOVICH, *Variational Analysis and Generalized Differentiation, I: Basic Theory, II: Applications*, Springer, Berlin, 2006.

[17] B. S. MORDUKHOVICH, *Variational Analysis and Applications*, Springer, Cham, Switzerland, 2018.

[18] S. M. ROBINSON, *Some continuity properties of polyhedral multifunctions*, Mathematical Programming at Oberwolfach (Proc. Conf., Math. Forschungsinstitut, Oberwolfach, 1979). Math. Programming Stud. No. **14** (1981), 206–214.

[19] R. T. ROCKAFELLAR and R. J-B. WETS, *Variational Analysis*, Springer, Berlin, 1998.